Light Source Accelerator Physics, Lecture 1

Introduction to Accelerator Physics of Storage Rings

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Outline

- Transverse optics
- Longitudinal optics
- Synchrotron Radiation

N.B. Lecture borrows heavily from David Robin's USPAS'03 & USPAS'07 lectures (www…)

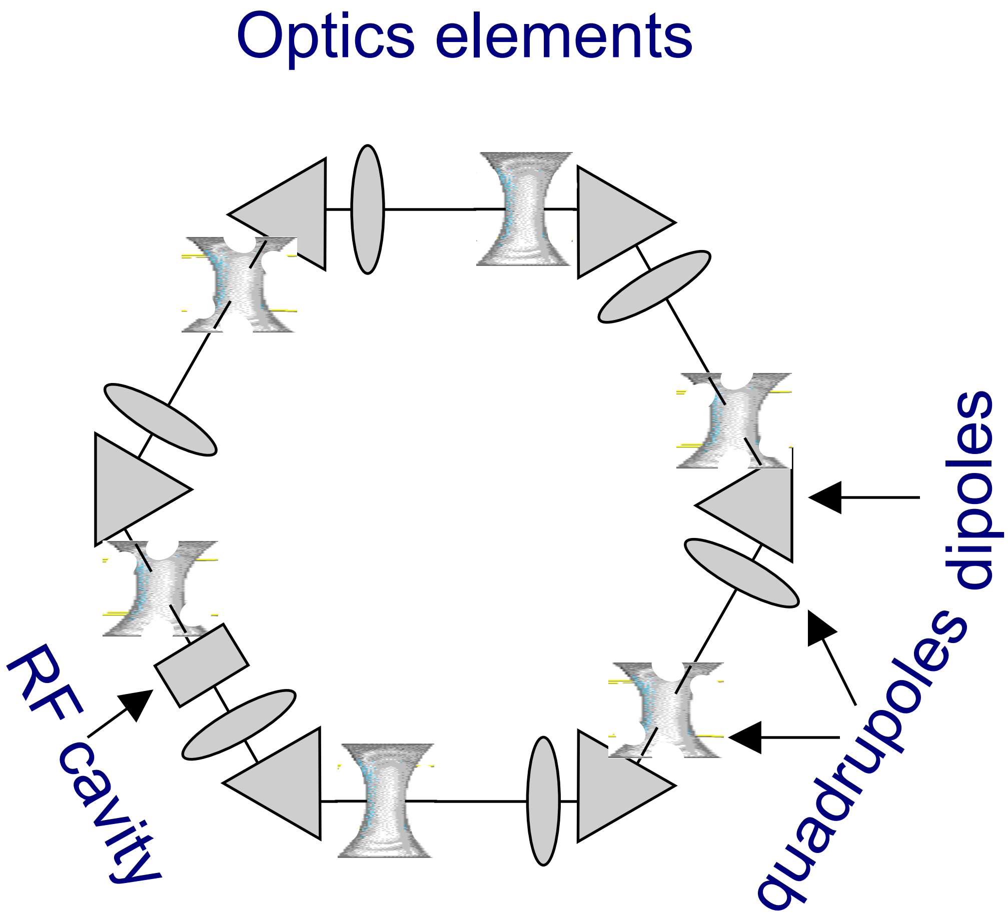
Concepts

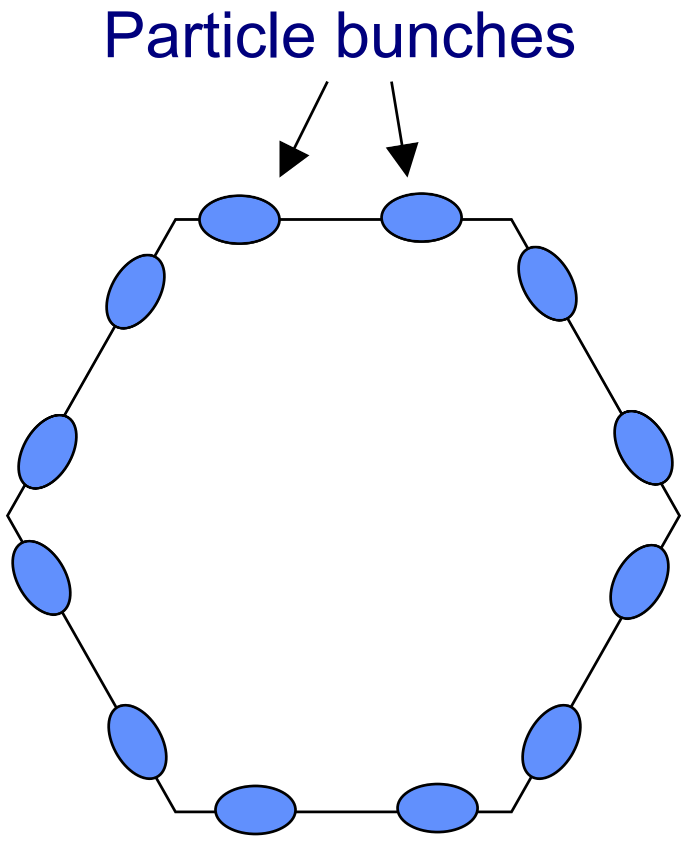
Want to touch on a number of concepts including:

- **Linear optics**
- **Equation of motion**
- **Transfer matrix**
- **Twiss parameters and phase advance**
- **Betatron tune**
- **Dispersion**
- **Momentum compaction**
- **Chromaticity**
- **Energy spread**
- **Emittance**

Particle Storage Rings

In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.

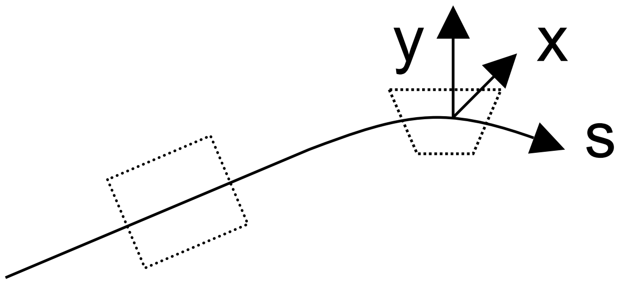




Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



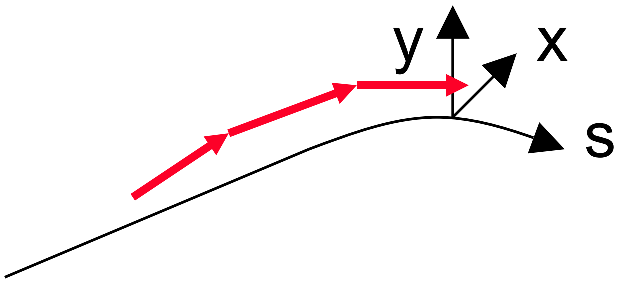
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$



Note sometimes one uses canonical momentum rather than x' and y'

Equations of Motion in a Storage Ring

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m **is the relativistic mass of the particle,**

e **is the charge of the particle,**

v **is the velocity of the particle,**

a **is the acceleration of the particle,**

E **is the electric field and,**

B **is the magnetic field.**

- **Lorentz force in practical units (B-field only)**

$$\mathrm{dx}' = \mathrm{ds}/\rho \quad \rightarrow \quad \mathrm{x}'' = 1/\rho$$

$$1/\rho[\mathrm{m}] = 0.3\,\mathrm{B(T)}/\mathrm{E[GeV]} \quad \text{or} \quad 1/\rho[\mathrm{m}] = \mathrm{B(T)}/(\mathrm{B}\rho) \;\; \text{where } \mathrm{B}\rho[\mathrm{Tm}] = \mathrm{E[GeV]}$$

Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

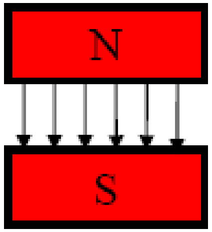
$$B_x = 0$$
$$B_y = B_o$$

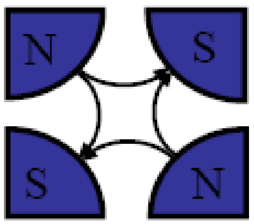
Quadrupoles → used for focusing

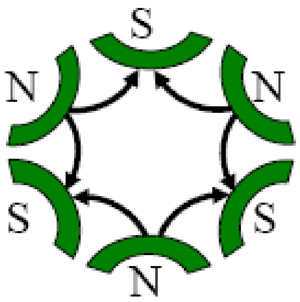
$$B_x = Ky$$
$$B_y = Kx$$

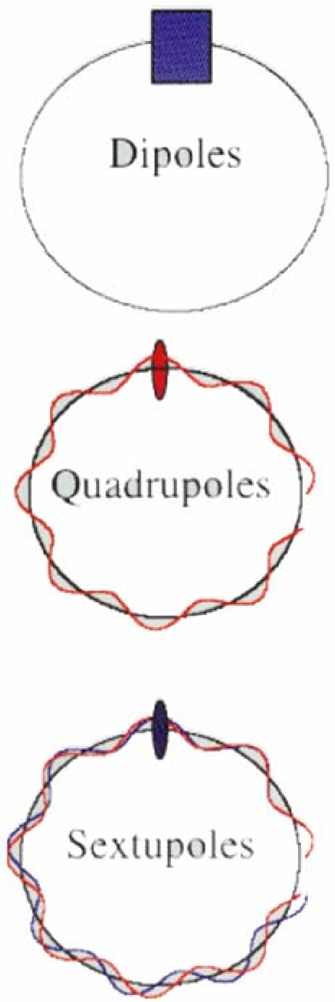
Sextupoles → used for chromatic correction

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$



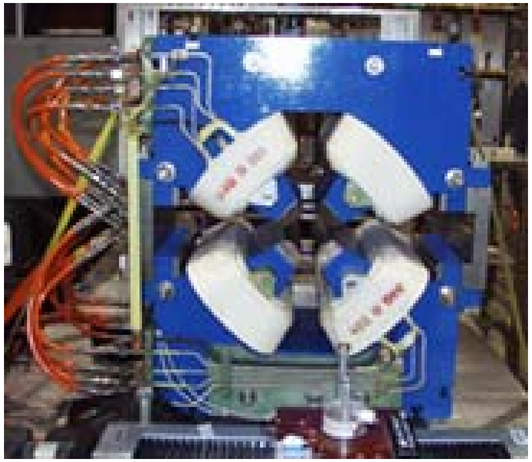






Magnets that make up a storage ring

Dipole (bends e^- beam)



Quadrupole (focuses e^- beam)

Sextupole (corrects aberrations)

Assembled magnet lattice

Equations of motion – Linear fields

- The equations of motion are:

$$
\begin{aligned}
x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right) x &= \frac{1}{\rho(s)} \frac{\Delta P}{P} \\
y'' + k(s)\, y &= 0
\end{aligned}
$$

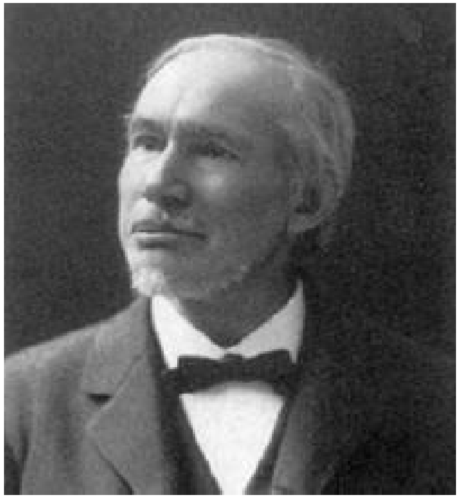
- Inhomogeneous equations with s-dependent coefficients
- Note that the term $1/\rho^2$ corresponds to the dipole weak focusing
- The term $\Delta p/(p\rho)$ is present for off-momentum particles.

Hill's equation

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$x'' + K_x(s)\ x = 0$$
$$y'' + K_y(s)\ y = 0$$

with
$$K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right)\ ,\ \ K_y(s) = k(s)$$


George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring or in transport line with symmetries, coefficients are periodic $K_x(s) = K_x(s+C)\ ,\ \ K_y(s) = K_y(s+C)$
- Not feasible to get analytical solutions for all accelerator

Two approaches

There are two approaches to introduce the motion of particles in a storage ring

1. The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, …

2. The way that our computer models actually do it – transfer matrices/linear algebra, plus transfer maps.

It is worthwhile to become proficient in both approaches.

Transfer matrix of a drift space

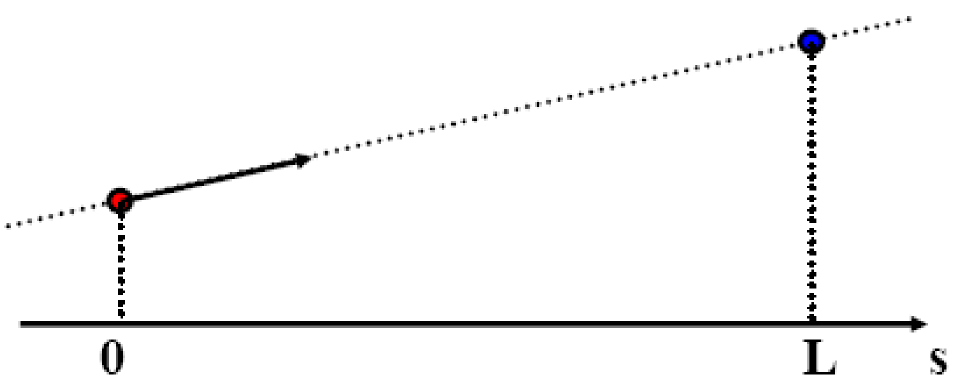
- Consider a drift (no magnetic elements) of length $L=s-s_0$

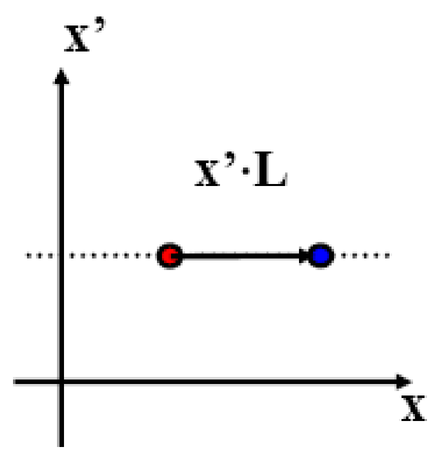
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\mathcal{M}_{\text{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} u(s) &= u_0 + (s-s_0)u'_0 = u_0 + Lu'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if there is a slope. Slope remains unchanged





Transfer matrix in a thin-lens quadrupole

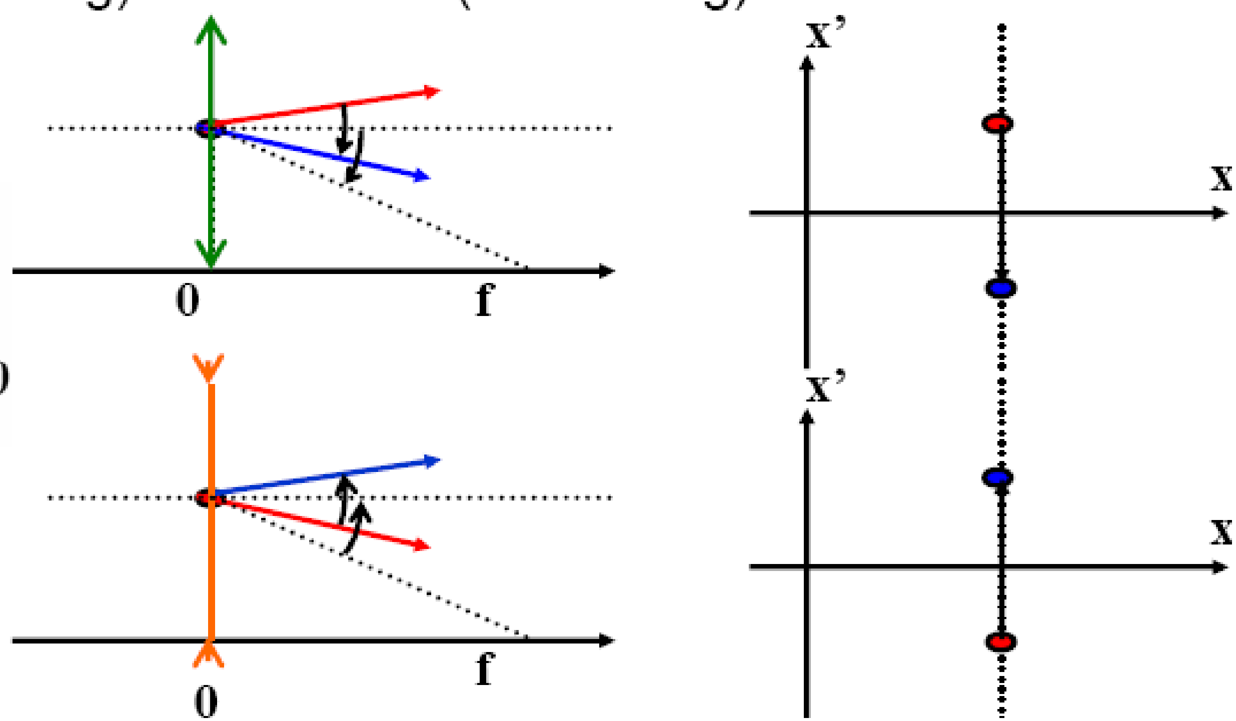
- Consider a lens with focal length ±f

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

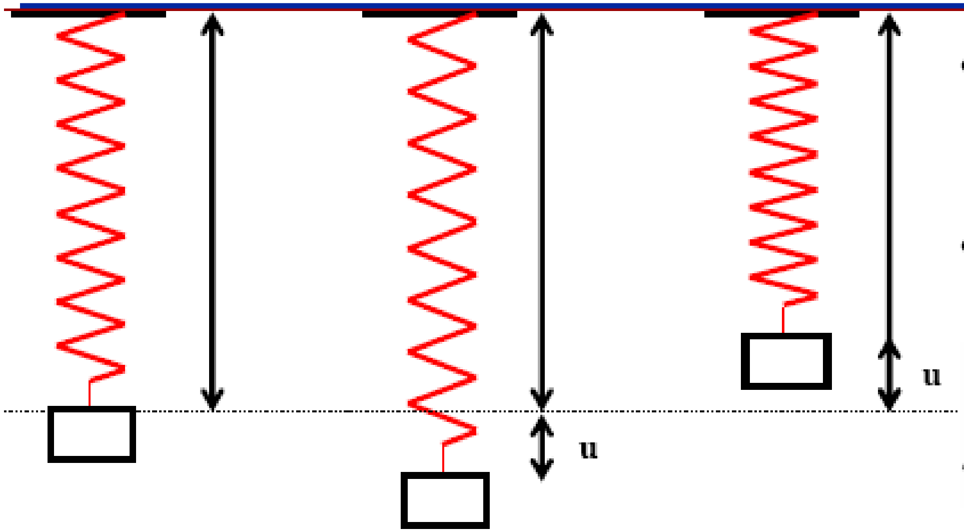
$$\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}$$

- Slope diminishes (focusing) or increases (defocusing). Position remains unchanged

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= u_0 \mp \frac{1}{f}u'_0 \end{aligned}$$



Motion in finite-length quadrupole = harmonic oscillator



- Consider K(s) = k_0 = constant

$$u'' + k_0 \; u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\;u(0) + S(s)\;u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

with

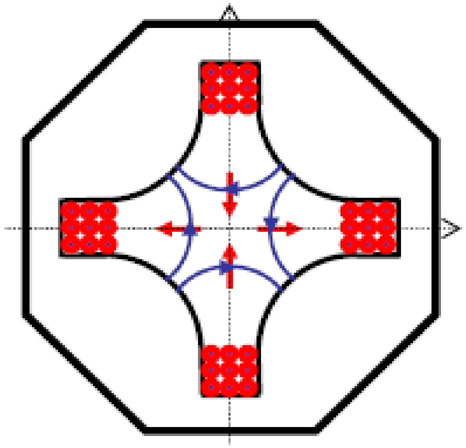
$$C(s) = \cos(\sqrt{k_0} s) \;, \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0} s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|} s) \;, \quad S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|} s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Transfer matrix of a finite-length quadrupole



- Consider a quadrupole magnet of length **L**. The field is

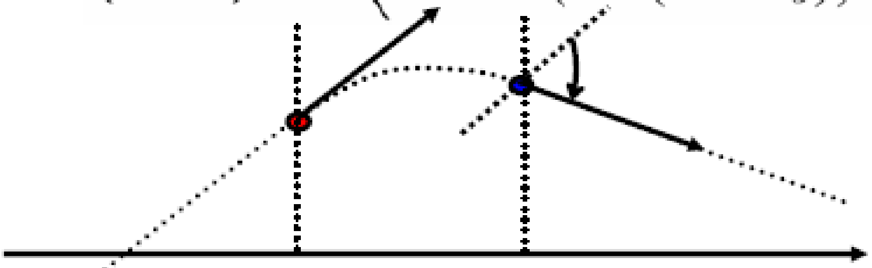
$$B_y = -G(s)x\ ,\quad B_x = -G(s)y$$

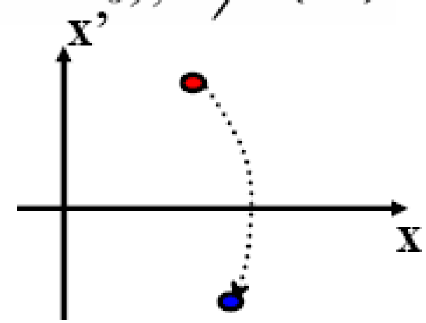
- with normalized quadrupole gradient (in $\mathbf{m}^{-2}$)

$$k = \frac{G}{B_0 \rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$





Sector Dipole

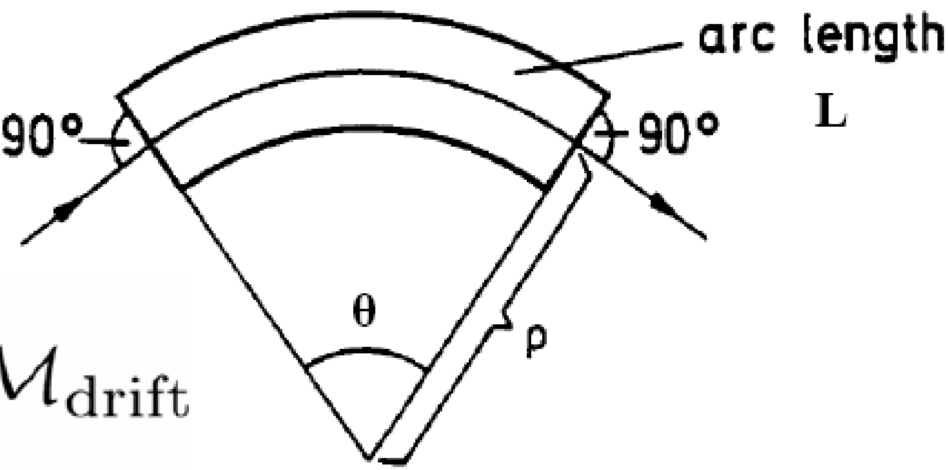
- Consider a dipole of length **L**. By setting in the focusing quadrupole matrix

$$k = \frac{1}{\rho^2} > 0$$

the transfer matrix for a sector dipole becomes

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

with a bending radius $\theta = \frac{L}{\rho}$

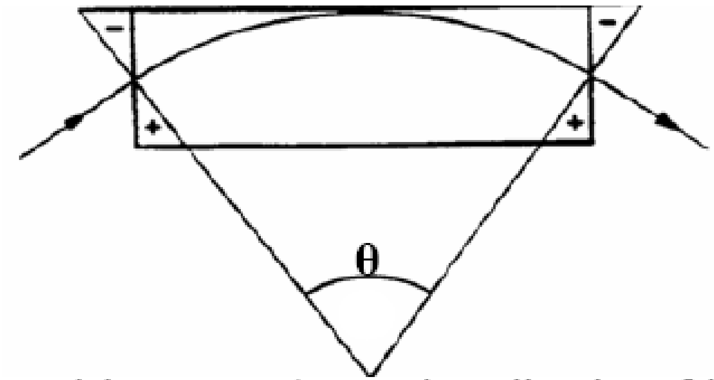


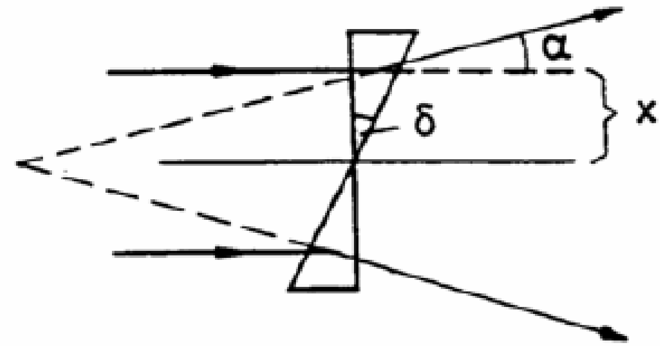
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane

Rectangular dipole





- Consider a rectangular dipole of length **L**. At each edge, the deflecting angle is

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho} \qquad \frac{1}{f} = \frac{\tan\delta}{\rho}$$

It acts as a thin defocusing lens with focal length

- The transfer matrix is $\mathcal{M}_{\mathrm{rect}} = \mathcal{M}_{\mathrm{edge}} \cdot \mathcal{M}_{\mathrm{sector}} \cdot \mathcal{M}_{\mathrm{edge}}$ with $\mathcal{M}_{\mathrm{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$
- For **θ<<1**, **δ=θ/2**.
- In deflecting plane (like **drift**) in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\mathrm{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\mathrm{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

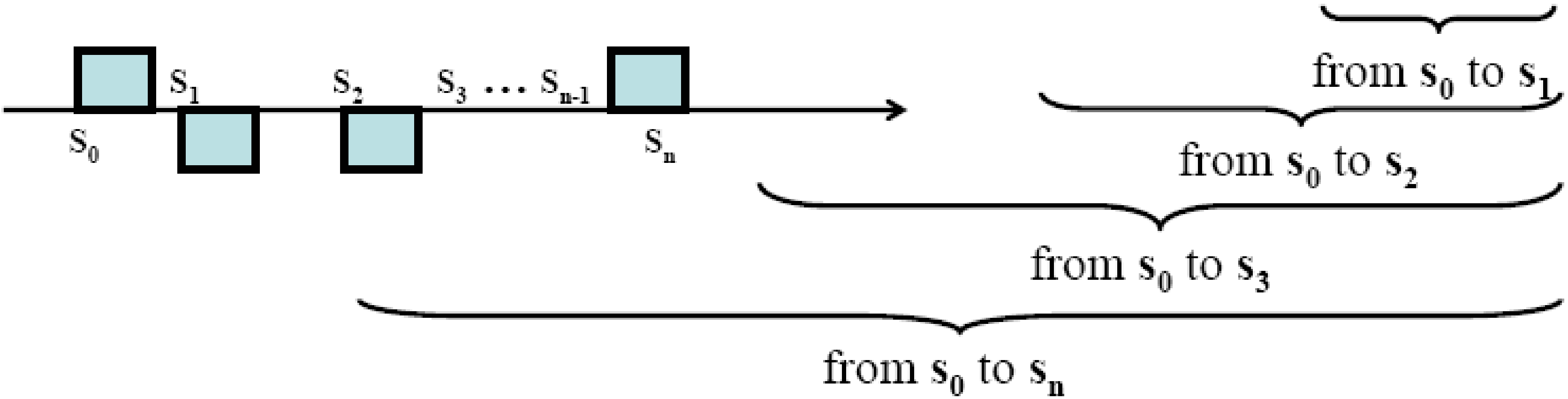
Transfer Matrix Formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$
  which is always true for conservative systems
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$
- The accelerator can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s0)$$



4x4 matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

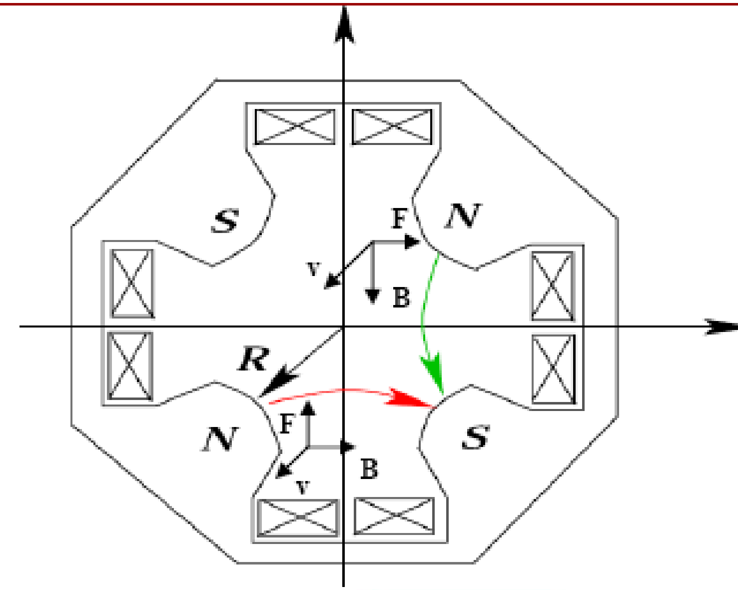
$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

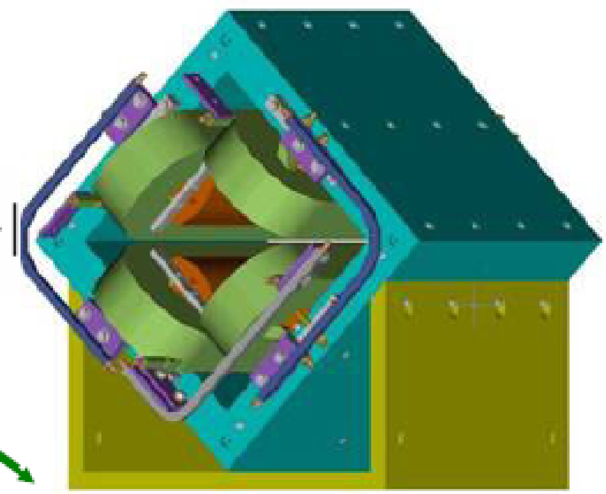
to get a total 4x4 matrix

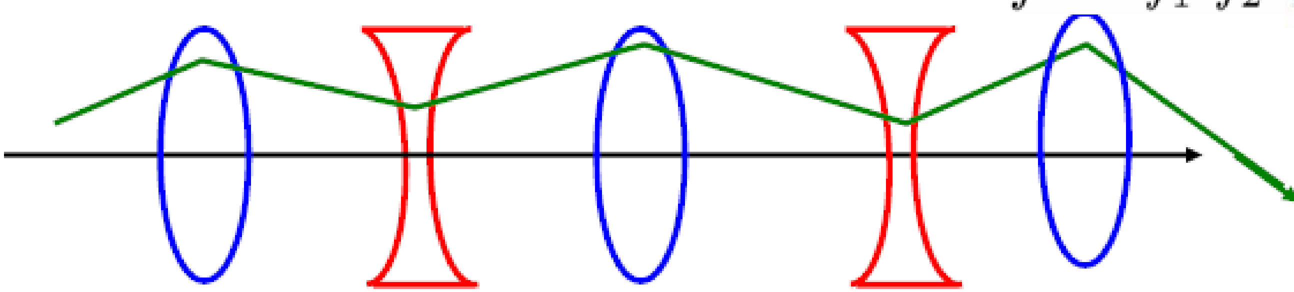
$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Quadrupoles

- Quadrupoles are focusing in one plane and defocusing in the other
- The field is $(B_x, B_y) = g(y, x)$
- The resulting force $(F_x, F_y) = k(y, -x)$
- Need to alternate focusing and defocusing in order to control the beam, i.e. **alternating gradient focusing**
- From optics we know that a combination of two lenses with focal lengths **f1** and **f2** separated by a distance **d** $$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\ f_2}$$
- If $f_1 = -f_2$, there is a net focusing effect, i.e. $\frac{1}{f} = \left|\frac{d}{f_1\ f_2}\right|$







Strong focusing

Weak Focusing

- V. Veksler and E. M. McMillan around 1945

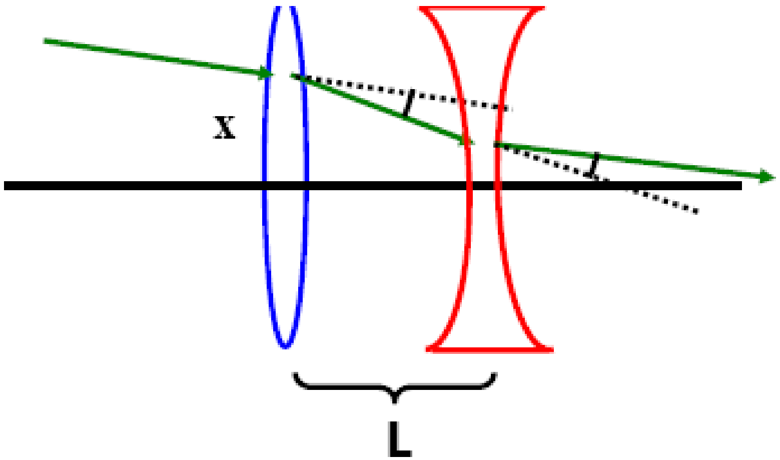
Strong Focusing

- Christofilos (1950), Courant, Livingston, and Snyder (1952)

Christofilos

Courant Livingston Snyder

Alternate Gradient Focusing



- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths $\mathbf{f_1}$ and $\mathbf{f_2}$ separated by a distance **L**.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\mathrm{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with the **total focal length**

$$\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\ f_2}$$

- Setting $\mathbf{f_1 = - f_2 = f}$ $\quad \frac{1}{f^\star} = \frac{L}{f^2}$
- Alternating gradient focusing seems overall focusing
- This is only valid in thin lens approximation!!!

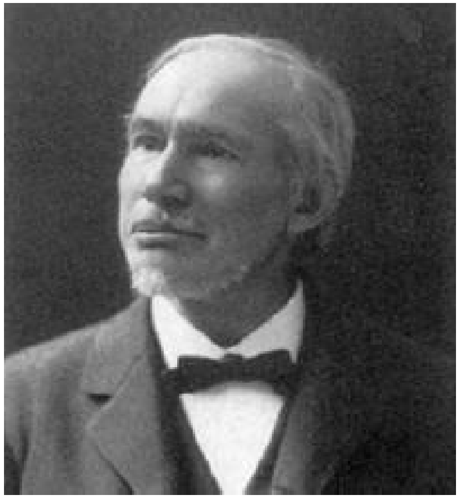
Back to Hill's equation …

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}$$

with

$$K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right)\ ,\quad K_y(s) = k(s)$$


George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring or in transport line with symmetries, coefficients are periodic $K_x(s) = K_x(s+C)\ ,\ \ K_y(s) = K_y(s+C)$
- Not feasible to get analytical solutions for all accelerator

Hills equation

The solution can be parameterized by a psuedo-harmonic oscillation of the form

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

where $\beta(s)$ is the beta function,

$\alpha(s)$ is the alpha function,

$\varphi_{x,y}(s)$ is the betatron phase, and

ε is an action variable

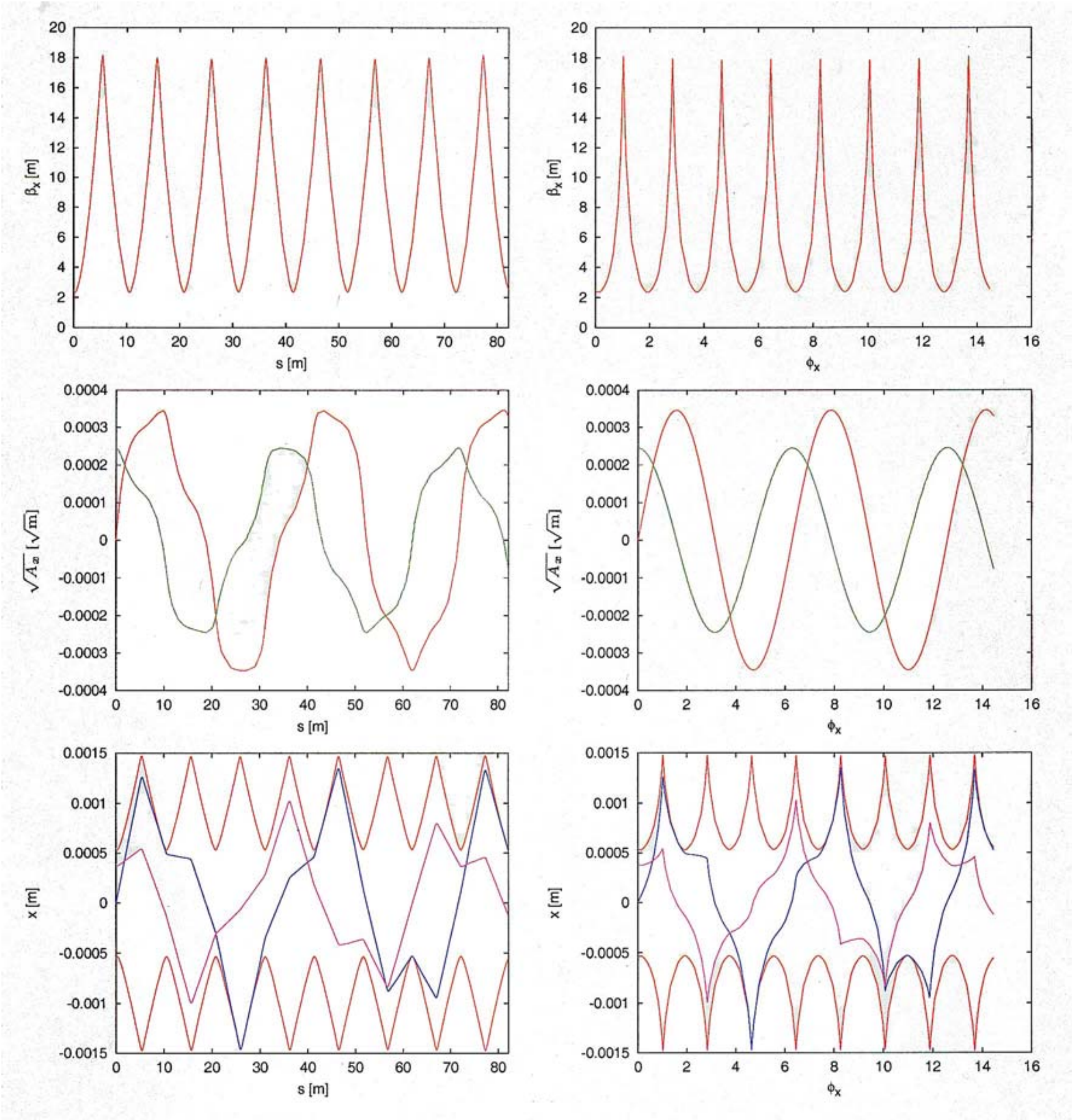
$$\alpha(s) \equiv -\frac{\beta'(s)}{2}$$

$$\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$$

$$\varphi = \int_0^s \frac{ds}{\beta}$$

& 2nd order differential equation for β(s).

Example of Twiss parameters and trajectories



Transfer matrices & β-functions

- Transfer matrices and β-functions came from same equation
- They are related:

$$R_{fi} = \begin{pmatrix} \sqrt{\frac{\beta_f}{\beta_i}}\left(\cos\varphi_{fi} + \alpha_i \sin\varphi_{fi}\right) & \sqrt{\beta_f \beta_i}\sin\varphi_{fi} \\ -\frac{1+\alpha_i\alpha_f}{\sqrt{\beta_f\beta_i}}\sin\varphi_{fi} + \frac{\alpha_i - \alpha_f}{\sqrt{\beta_f\beta_i}}\cos\varphi_{fi} & \sqrt{\frac{\beta_i}{\beta_f}}\left(\cos\varphi_{fi} - \alpha_f \sin\varphi_{fi}\right) \end{pmatrix}$$

- One-turn transfer map (*i*=*f*)

$$R_{one-turn} = \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

- Tune is defined as number β-oscillations/turn, $\nu = \phi_{\text{one-turn}}/2\pi$

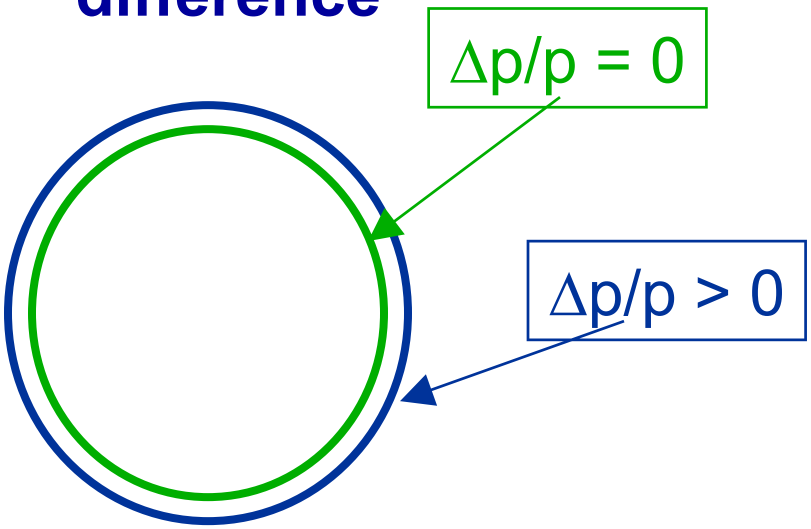
$$\nu = \frac{1}{2\pi}\cos^{-1}\left(\frac{Tr(R)}{2}\right), \qquad \beta = \frac{R_{12}}{\sin(2\pi\nu)}, \qquad \alpha = \frac{R_{11} - R_{22}}{2\sin(2\pi\nu)}$$

- This is how a computer calculates optics functions – using transfer matrices

Dispersion and momentum compaction

Assume that the energy is fixed → no cavity or damping

- **Find the closed orbit for a particle with slightly different energy than the nominal particle. The dispersion is the difference in closed orbit between them normalized by the relative momentum difference**

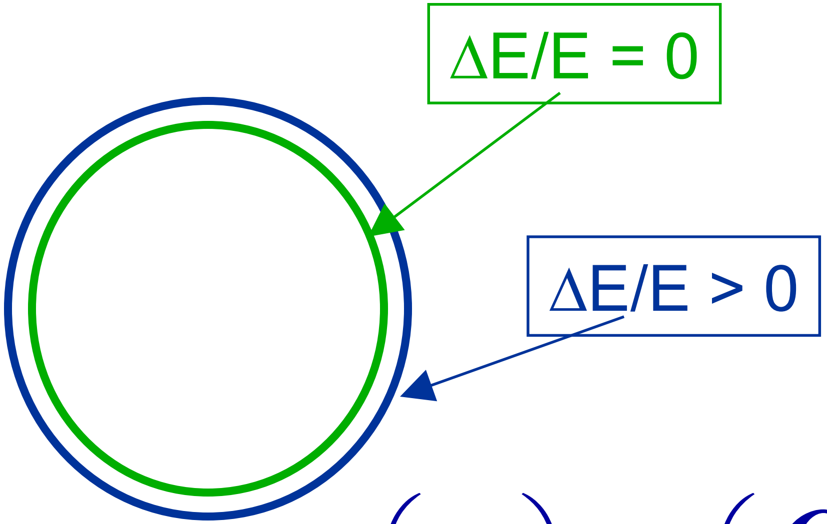


$$x = D_x \frac{\Delta p}{p}, y = D_y \frac{\Delta p}{p}$$

$$x' = D'_x \frac{\Delta p}{p}, y' = D'_y \frac{\Delta p}{p}$$

Dispersion

Dispersion, *D*, is the change in closed orbit as a function of energy

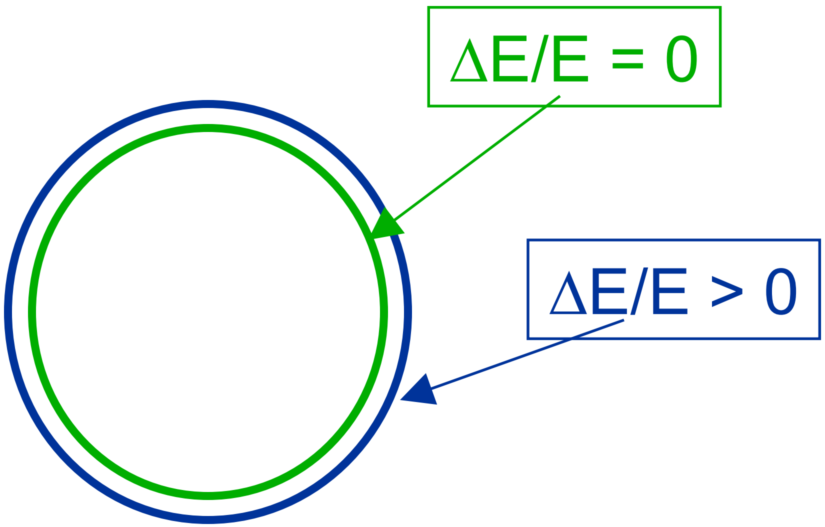


$$x = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_i$$

Momentum Compaction

Momentum compaction, α, is the change in the closed orbit length as a function of energy.

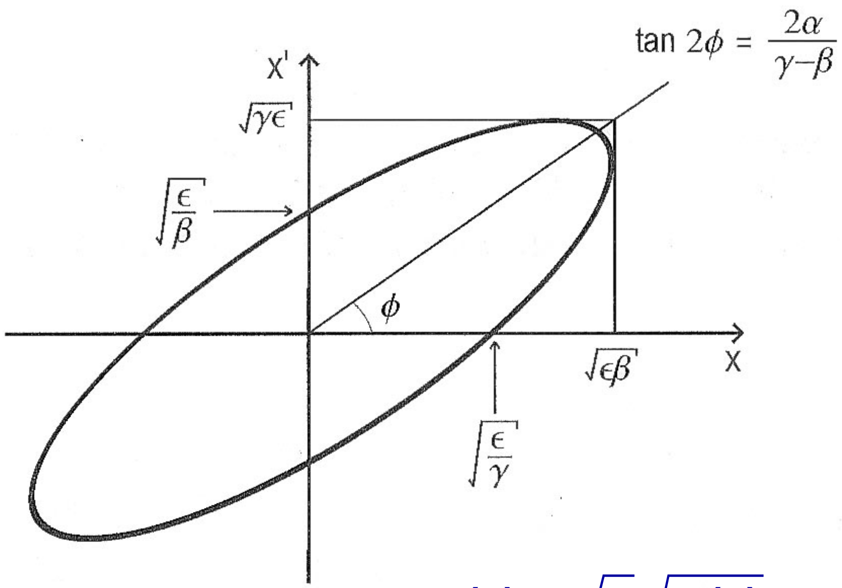


$$\frac{\Delta L}{L} = \alpha \frac{\Delta E}{E}$$

$$\alpha = \int_0^{L_0} \frac{D_x}{\rho} ds$$

Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ε:

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\,\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\,\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

Emittance Definition/Statistical

- Emittance defined as the phase space area occupied by an ensemble of particles
- Phase space means consisting of pairs of position and (canonical) momentum variables
- Example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).
- Emittance can be defined as a statistical quantity (beam is composed of finite number of particles)

$$\varepsilon_{geometric,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

- In certain systems (I will not go into the mathematical details) the (normalized) emittance is a conserved quantity – e.g. single charged particle traveling down a magnetic structure – Liouville.

Transport of the beam ellipse

Beam ellipse matrix

$$\sum_{beam}^{x} = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum_{beam,f}^{x} = R_{x,i-f} \sum_{beam,i}^{x} R_{x,i-f}^{T}$$

Transport of the beam ellipse

Transport of the twiss parameters in terms of the transfer matrix elements

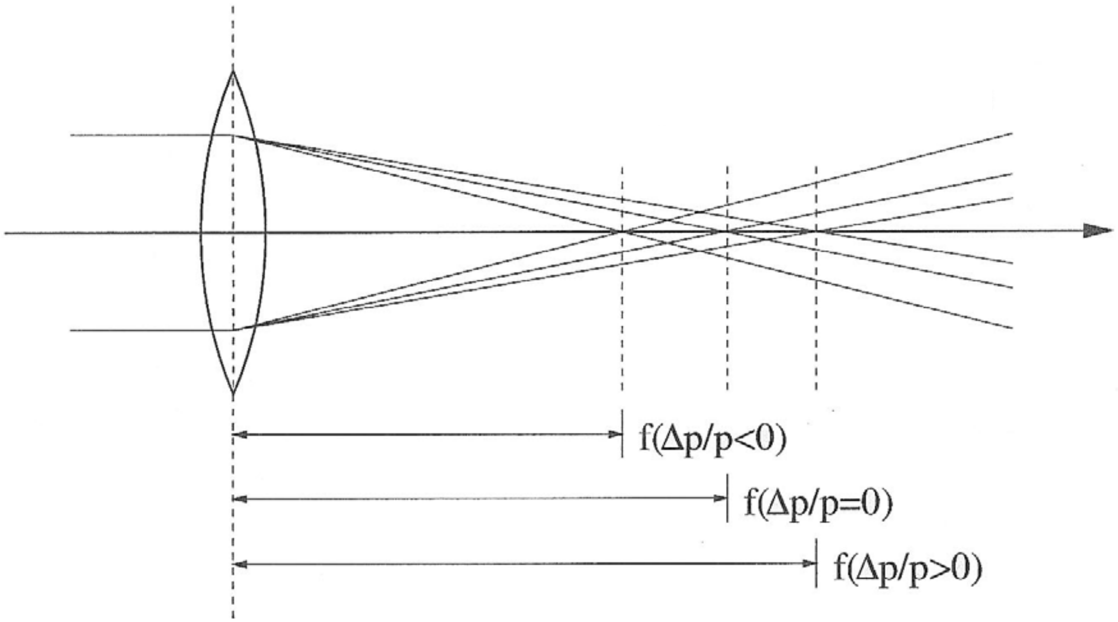
$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_f = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & 1+C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix}\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_i$$

Transfer matrix can be expressed in terms of the twiss parameters and phase advances

$$R_{fi} = \begin{pmatrix} \sqrt{\dfrac{\beta_f}{\beta_i}}\left(\cos\varphi_{fi} + \alpha_i \sin\varphi_{fi}\right) & \sqrt{\beta_f\beta_i}\,\sin\varphi_{fi} \\ -\dfrac{1+\alpha_i\alpha_f}{\sqrt{\beta_f\beta_i}}\sin\varphi_{fi} + \dfrac{\alpha_i - \alpha_f}{\sqrt{\beta_f\beta_i}}\cos\varphi_{fi} & \sqrt{\dfrac{\beta_i}{\beta_f}}\left(\cos\varphi_{fi} - \alpha_f \sin\varphi_{fi}\right) \end{pmatrix}$$

Chromatic Aberration

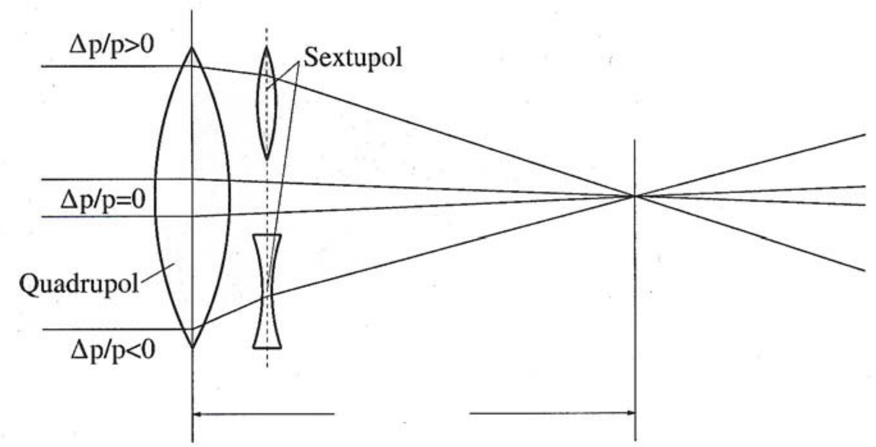
Focal length of the lens is dependent upon energy



Larger energy particles have longer focal lengths

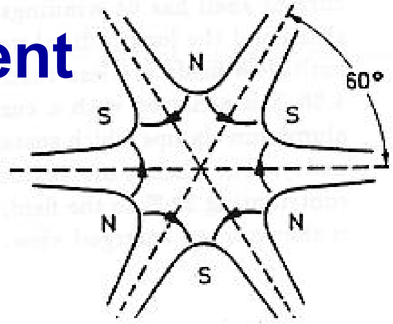
Chromatic Aberration Correction

By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent Quadrupole

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$



Chromatic Aberration Correction

Chromaticity, ν', is the change in the tune with energy

$$\nu' = \frac{d\nu}{d\delta}$$

Sextupoles can change the chromaticity

$$\Delta\nu_x' = \frac{1}{2\pi}\left(\Delta S \beta_x D_x\right)$$

$$\Delta\nu_y' = -\frac{1}{2\pi}\left(\Delta S \beta_y D_x\right)$$

where

$$\Delta S = \left(\partial^2 B_y \big/ \partial x^2\right) \mathbf{length} \Big/ (2B\rho)$$