Lecture 2
Aspects of Transverse Beam Dynamics

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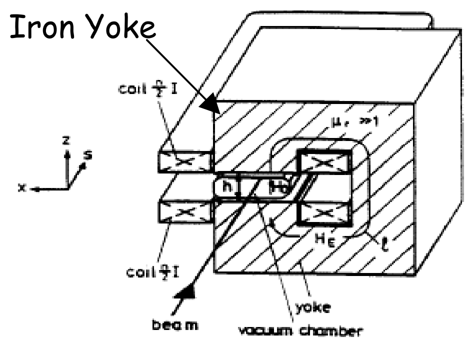
Accelerator and Beamline Magnets

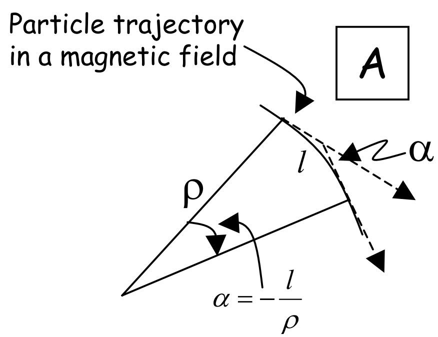
Dipole Magnet: Dipole magnet is a device used to bend the path of charged particles during beam transport. The radius of curvature of a charged particle in a constant magnetic field perpendicular to its path is,



$$\frac{1}{R} = \frac{1}{\rho}[\mathrm{m}^{-1}] = \frac{eB_0}{p} = \frac{0.2998\,B[T]}{p[GeV/c]}\;;\mathrm{B}_0 = \frac{0.04\,I_{total}[amp].n}{h[cm]} \qquad (1)$$

n=number of turns h=pole gap

Quadrupole Magnet: A device used to focus charged particle beam during beam transport.



Let us see what is the relationship between focal length, f, and the quadrupole strength. Fig. A shows bending of a charged particle in a magnetic field perpendicular to the plane of the paper and "B" shows optical analogue of focusing. Then the deflection angle,

$$\alpha = -\frac{l}{\rho} = -\frac{r}{f} = \frac{eB_\phi}{p}l = -\frac{eB_\phi}{\beta E}l \qquad (2)$$

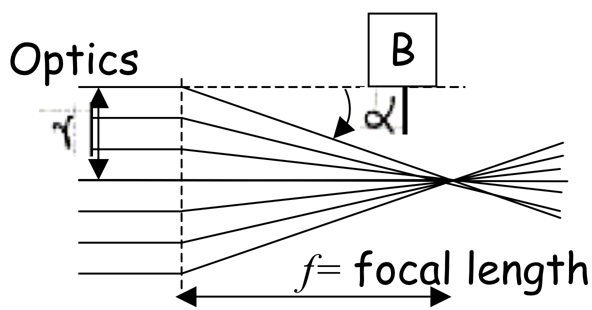
But the total bending field B_ϕ is given by,

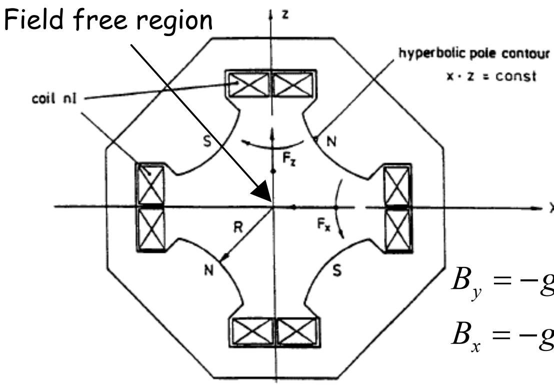
$$B_\phi = \frac{dB_\phi}{dr}r = gr$$

Then,

$$\alpha = -\frac{egrl}{\beta E} = -\frac{r}{f} \quad \text{or} \quad \frac{1}{f} = kl;\; k = \frac{eg}{\beta E} = \frac{eg}{p} \qquad (3)$$

Quad strength





$$B_y = -gx$$
$$B_x = -gy$$

$$k[m^{-2}] = \frac{0.2998\, g[Tesla/m]}{\beta E[GeV/c]}; \quad g = \frac{2\mu_0 nI}{R^2} \qquad (4)$$

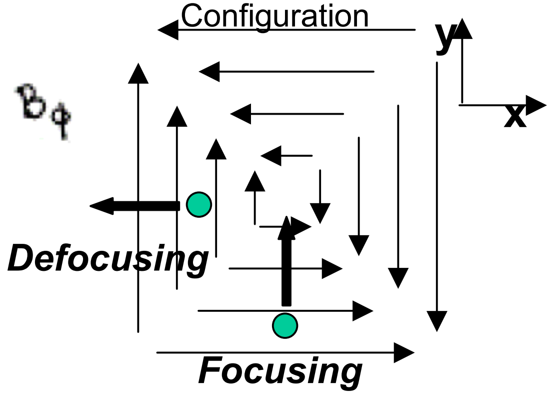
The quadrupole magnets provide material free aperture and focusing.

A conventional quadrupole magnet used in synchrotrons has four iron poles with hyperbolic contours.

Interesting features:
The horizontal force component depends only on the horizontal position of the particle trajectory. Similarly for the vertical axis.

$$F_x = -evgx \quad \text{and} \quad F_y = evgy \qquad (4)$$

Quadrupole Field Configuration

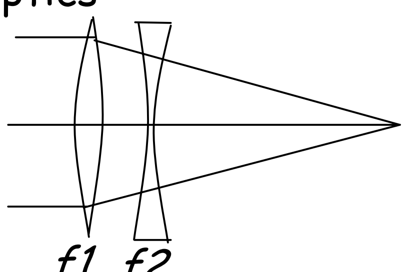


A single "quad" can provide focusing only in one plane because of the magnetic field configuration. Hence, one needs at least two consecutive quadrupole magnets to get overall focusing of all charged particles.

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2} \qquad (5)$$

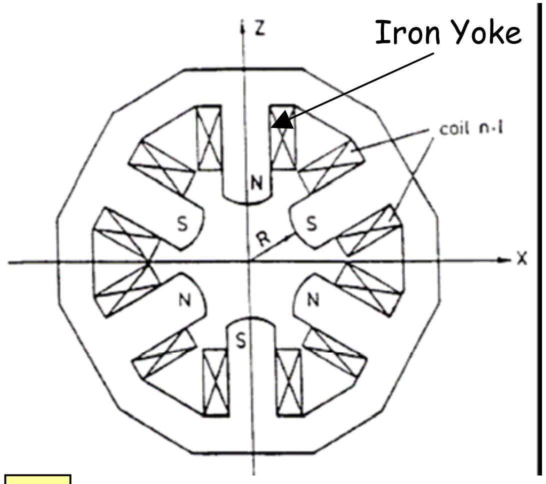
f_1= focal length of 1st quad
f_2= focal length of 2nd quad
L = separation between two quads

Optics



Linear Machine: Contains only dipoles and quadrupoles. In these machines the horizontal and vertical motions are "completely decoupled"

Sextupole Magnet: These are used for chromatic corrections during beam transport, storing or beam acceleration.



The sextupole magnets generate non-linear magnetic field and introduce horizontal and vertical motion of the beam

$$B_y = \frac{1}{2} g'(x^2 - y^2),\ B_x = g' xy$$

$$g' = 6\mu_0 nI / R^3\ [\mathrm{m}^{-1}]$$

The momentum-independent sextupole strength is given by,

$$m = \frac{eg'}{p},\ \ m[m^{-3}] = \frac{0.2998\, g'[Tesla/m^2]}{p[GeV/c]} \qquad (6)$$

Multipole Field Expansion: General multipole field expansion is given by,

$$\begin{cases} B_x = gy + sxy + \frac{1}{6} o(3x^2 y - y^3) + \ldots & \text{(Vertical bending field)} \\ B_y = B_{0y} + gx + \frac{1}{2} s(x^2 - y^2) + \ldots & \text{(Horizontal bending field)} \end{cases} \qquad (7)$$

In an un-coupled case,

$$B_x = 0 \ \text{and}\ B_y = B_{0y} + gx + \frac{1}{2} sx^2 + \frac{1}{6} sx^3 + \ldots$$

g, s, o are quadrupole, sextupole and octupole strength parameters, respectively

Thus,

$$\frac{\mathrm{e}B_y}{p} = \frac{\mathrm{e}B_{0y}}{p} + \frac{\mathrm{e}g}{p} x + \frac{\mathrm{es}}{2p} x^2 + \ldots \qquad (8)$$

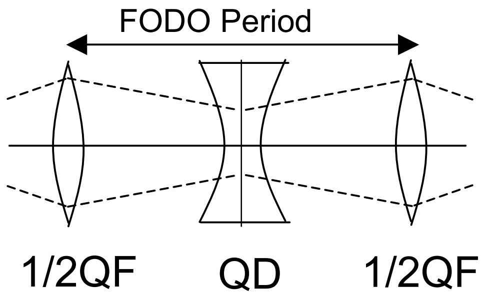
Dipole Quadrupole Sextupole contributions

Beam Transport and FODO Lattice

A beam transport comprises of magnetic elements that form a lattice which guides the charged particle beam from one point to another. Such a beam generally traverses through a vacuum beam pipe.

Lattice may be for

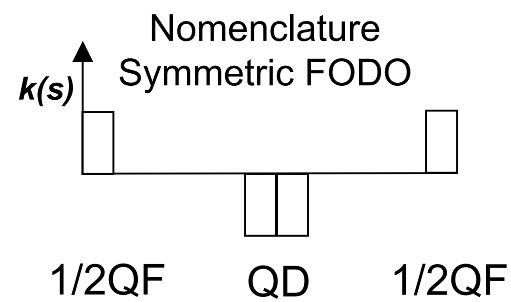
- Beam transfer line
- Circular beam storage ring or accelerator
- Linear accelerator



FODO Lattice:

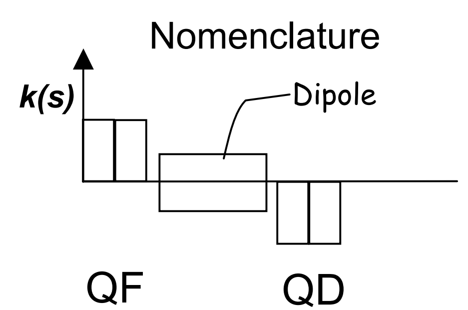
A lattice comprising symmetric quadrupole triplets that can focus beam in both X and Y planes or

a lattice comprising of Focusing-Drift-Defocusing-Drift with quad strengths.



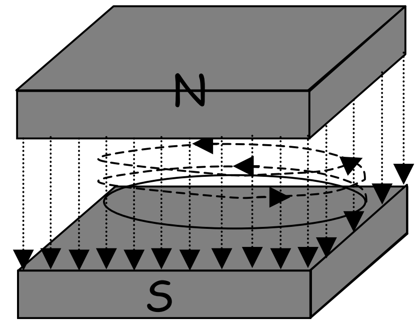
Beamline and Circular Machine:

FODO lattice can be repeated as many times as needed to form a beamline or a circular machine.



Weak Focusing Accelerators

So far we have assumed that the trajectory of the path of any particle in a beam is always perpendicular to the dipole magnetic field in a circular accelerator. Suppose a particle has a vertical angle $\neq 0$. Then the particle follows a spiral path and gets lost. Therefore we need an additional focusing force to keep the particle stable.



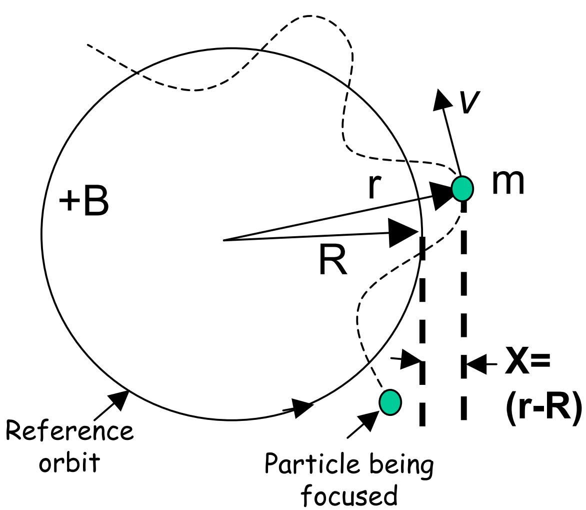
We know that for an ideal orbit

$$\frac{mv^2}{R} = qvB_y \Rightarrow \text{Restoring Force} = \frac{mv^2}{R} - qvB_y = 0 \qquad \boxed{9}$$

For any other particle the restoring force is given by,

$$F_x = \frac{mv^2}{R} - qvB_y \qquad \boxed{10}$$

To keep the beam particle focused one needs the gradient component in the magnetic field.



$$B_y = B_{0y} + \frac{dB_y}{dx}x + \ldots = B_{0y}\left(1 + \frac{R}{B_{0y}}\frac{dB_y}{dx}\frac{x}{R} + \ldots\right)$$

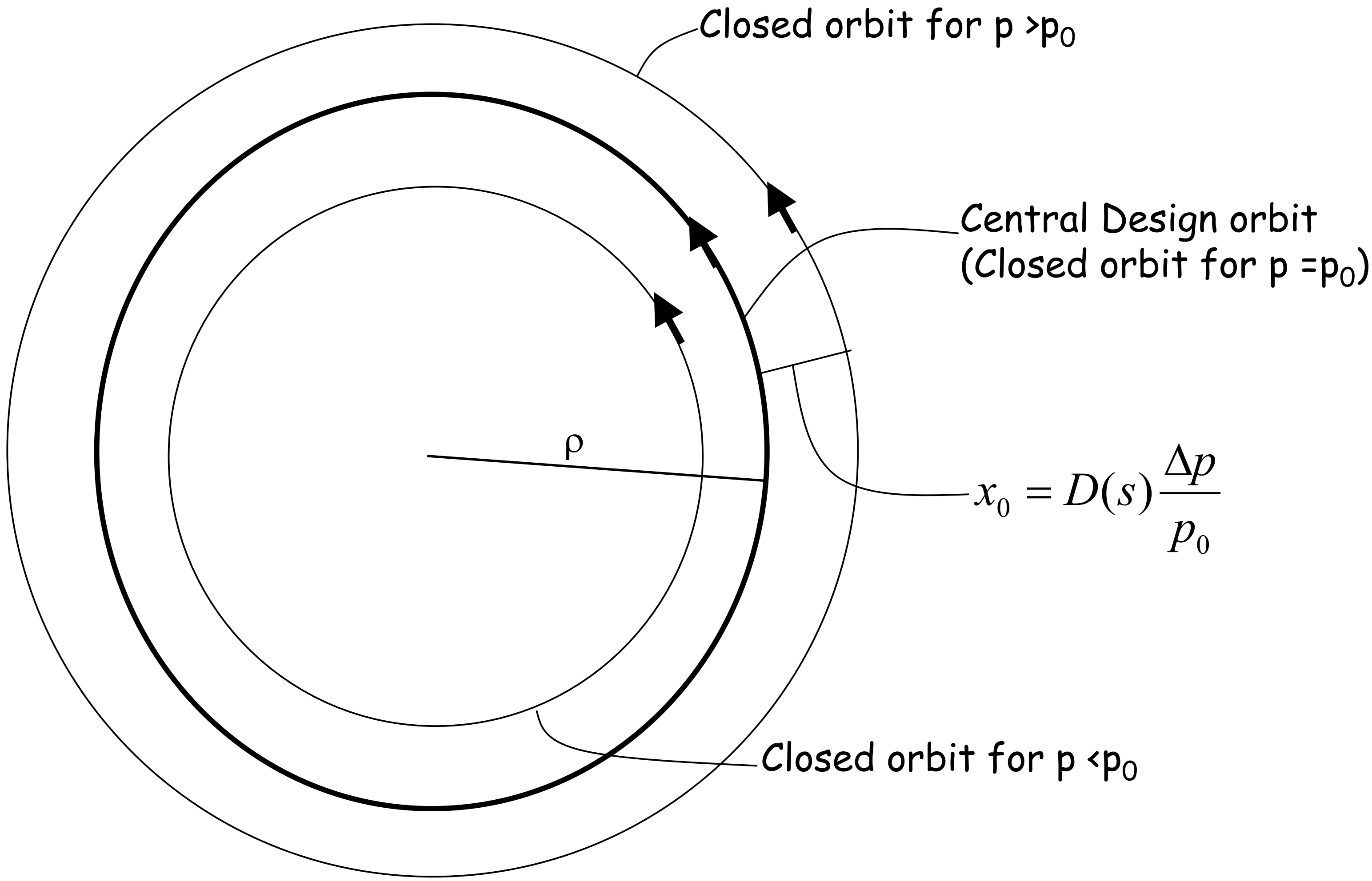
$$\approx B_{0y}\left[1 - n\frac{x}{R}\right] \quad \text{with } n = -\frac{R}{B_{0y}}\frac{dB_y}{dx} \Leftarrow \text{Field Index} \qquad \boxed{11}$$

Then one can show that if 0<n<1 ("**Steenbeck's criterion**") particle can be focused in both x and y planes and beam can be made stable.

A circular accelerator with this stability criterion is called a "Weak Focusing Accelerator"

Weak Focusing Accelerators (cont.)

An optical analogue of weak focusing accelerator is shown here



Strong Focusing Accelerators

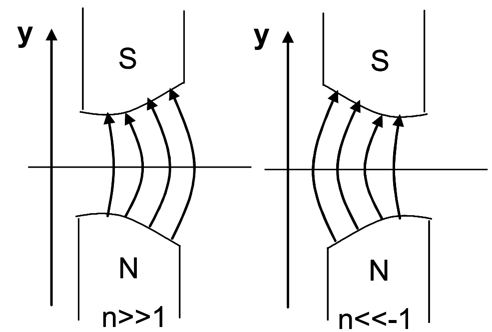
The circular accelerators with magnetic field gradients

$$n << -1 \text{ and } n >> 1$$ 12

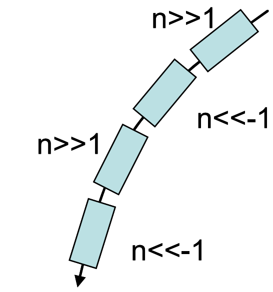
used alternatively also provide very high stability to the beam. These are called "Strong Focusing Accelerators"

Christofilos (1950)

Courant, Livingston and Snyder (1952)



Combined function magnets



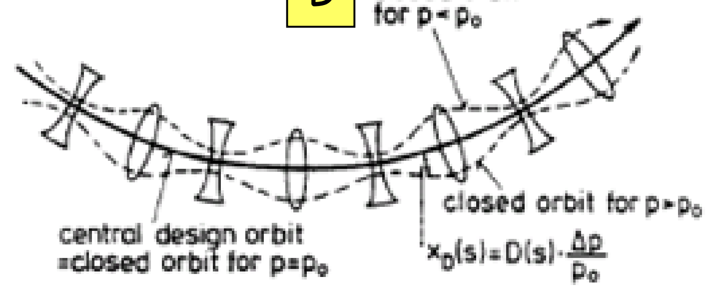
Alternating gradient focusing

Combination of focusing and de-focusing quads shown in Fig. B with $f_1 = -f_2$ will give

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2} = \frac{L}{f_1^2}$$ $\therefore$ Always focusing

Optical analogue of such an accelerator with the combination of focusing and de-focusing quads isshown in Fig. B .

B



Closed Orbit for the strong focusing lattice

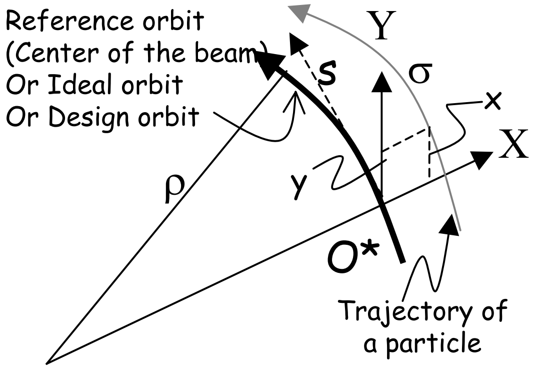
Earlier strong focusing accelerators are built with combined function magnets. Recent strong focusing accelerators used separated function magnets;

- Dipoles for bending
- Quads for focusing
- Sextupoles for chromatic corrections

Accelerators with combined function magnets

CERN PS	n≈±288
Fermilab Booster	n=165, -207
Fermilab Recycler	n=620, -598

Coordinate System



We use an orthogonal right handed coordinate system (y,x,s) that follows the reference orbit particle.

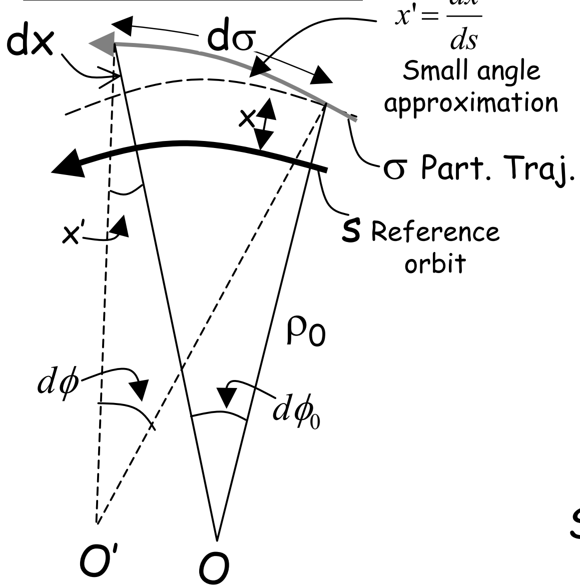
∴ This is a moving coordinate system with

- x, y – deviation of particle trajectory from reference orbit at the point of interest "O* "
- s – tangential vector at "O* "
- σ - individual particle trajectory.

This will be our coordinate system in the rest of our analysis

Hill's Equation:

Horizontal Plane



From this picture,

$$(d\sigma)^2 = [d\phi_0(\rho_0 + x)]^2 + (dx)^2 \approx (ds)^2\left[1 + \frac{x}{\rho_0}\right]^2; \quad \rightarrow \quad d\phi_0 \rho_0 = ds$$

$$\text{or} \quad d\sigma = ds\left[1 + \frac{x}{\rho_0}\right] \qquad \text{(13)}$$

The equation of motion is going to be with respect to the reference orbit.

Note that in the deflection plane

$$dx' = d\phi_0 - d\phi \Rightarrow x'' = \frac{d^2x}{ds^2} = \frac{d\phi_0}{ds} - \frac{d\phi}{ds} \qquad \text{(14)}$$

Substituting

$$d\phi_0 = \frac{ds}{\rho_0}, \quad d\phi = \frac{d\sigma}{\rho} = \frac{ds}{\rho}\left[1 + \frac{x}{\rho_0}\right] \qquad \text{(15)}$$

We get the equation of motion as,

$$x'' = \frac{1}{\rho_0} - \frac{1}{\rho}\left[1 + \frac{x}{\rho_0}\right] \qquad \text{16}$$

For the monochromatic beam with momentum p_0, the curvature of the charged particle in an electromagnetic field is given by,

$$\frac{1}{\rho} = \frac{qB_y}{p_0} = \frac{q}{p_0}\left[B_{0y} + gx + \frac{s}{2}(x^2 - y^2) + \dots\right]$$

$$\approx \left[\frac{1}{\rho_0} + kx\right] \qquad \text{17}$$

Then,

$$x'' = \frac{1}{\rho_0} - \left[\frac{1}{\rho_0} + kx\right]\left[1 + \frac{x}{\rho_0}\right] = -\left[k + \frac{1}{\rho_0^2}\right]x - \frac{k}{\rho_0}x^2$$

← very small so we can neglect

Thus, the equation of motion in horizontal plane becomes

$$x'' + Kx = 0; \quad \text{with} \quad K = \left[k + \frac{1}{\rho_0^2}\right] \qquad \text{18a}$$

Similarly in vertical plane

$$x'' - kx = 0; \quad \text{with} \quad k = \frac{q}{p_0}\frac{dB_{0x}}{dy}$$

Notice that this equation a) has not got dipole strength, b) change in sign **18b**

The term $\frac{x}{\rho_0^2}$ in the above equation describes the "**weak focusing**" of a bending magnet. In a large accelerator this term can be neglected.

Momentum Dispersion:

In reality, the particles in a beam are not monochromatic. Now let us look at a particle that has a small momentum offset, i.e.,

$$p \to p_0 + \Delta p = p_0(1+\delta); \quad \delta = \frac{\Delta p}{p_0} \qquad (19)$$

Then the curvature is rewritten as,

$$\frac{1}{\rho} = \frac{eB_y}{p_0}(1+\delta)^{-1} = \frac{e}{p_0}\left[B_{0y} + gx + \ldots\right]\left[1 - \delta + \delta^2 - \ldots\right]$$

$$\approx \left[\frac{1}{\rho_0} + kx - \frac{\delta}{\rho_0}\right] \qquad (20)$$

Thus, the equations of motion become

$$x'' + Kx = \frac{\delta}{\rho_0}$$

$$y'' + ky = \frac{\delta}{\rho_{y0}}$$

One can combine these two into one equation as,

$$u'' + K(s)u = \frac{\delta}{\rho_0} \qquad (21)$$

This has bending term $\rho(s)$ and focusing term $k(s)$. $K(s)$ is called spring constant

These are equations of motion for "strong focusing" charged particle beam accelerators (and for the beam transport lines). This equation is called "**Hill's equation**".

Notice that the magnitude of the focusing strength is a free parameter.

Solutions for the Equation of Motion (piecewise method)

Let us solve the simpler case of Hill's equations for a monochromatic beam., i.e., δ=0.

$$u''+K(s)u = 0 \qquad \text{22}$$

The principal solution to this equation (assuming K is a constant) are,

$$\text{For} \quad K > 0 : C(s) = \cos(\sqrt{K}s) \;\; \& \;\; S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s)$$

$$\text{For} \quad K < 0 : C(s) = \cosh(\sqrt{|K|}s) \;\; \& \;\; S(s) = \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \qquad \text{23}$$

which are linearly independent. "C"= cosine like function and "S"= sine like function. They satisfy initial conditions at $s=0$

$$C(0) = 1, C'(0) = 0, \;\; S(0) = 0 \;\; S'(0) = 1 \qquad \text{24}$$

Any arbitrary solution can be written as a linear combination of C and S

$$u(s) = C(s)u_0 + S(s)u'_0$$
$$u'(s) = C'(s)u_0 + S'(s)u'_0$$

with u_0, u'_0 as arbitrary initial coordinates of the particle trajectory — 25

In matrix form this is written as,

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix} = M \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix} \qquad \text{26}$$

The determinant $=CS'-C'S$ and its derivative, $CS''+C'S'-C'S'-CS''=CS''-C''S=0$ are independent of s

Solutions for the Equation of Motion (cont.)

For the initial conditions of $s=0$, the 2x2 matrix becomes,

$$\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix}_{s=0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

unit matrix — 27

Further, with negligible dissipating forces we find that, for any arbitrary beamline, the determinant

$$\det M = \begin{vmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{vmatrix} = 1$$

— 28

Remark: There are some cases where the above determinant $\neq 1$. This means there might be some accelerating/decelerating or quantum effect to be taken into account.

$$\begin{aligned} &\text{If} \quad \det M < 1 \quad \text{Damping} \\ &\qquad\qquad\quad > 1 \quad \text{Anti - damping} \end{aligned}$$

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Transformation Matrices for the Accelerator Components

Here we deal with some commonly used accelerator components

A: <u>Pure focusing quadrupole</u>: $\frac{1}{\rho_0} = 0, \quad k > 0, \quad s = l.$ Set $\theta = \sqrt{k}l = \sqrt{k}(s - s_0)$. Then,

$$M_{QF} = \begin{bmatrix} \cos(\theta) & \frac{\sin(\theta)}{\sqrt{k}} \\ -\sqrt{k}\sin(\theta) & \cos(\theta) \end{bmatrix};$$

For a thin lens, $l \to 0$, $M_{QF} = \begin{bmatrix} 1 & 0 \\ -kl & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$ — 30

B: <u>Pure defocusing quadrupole</u>: $\frac{1}{\rho_0} = 0, \quad k < 0, \quad s = l$. Set $\theta = \sqrt{|k|}l = \sqrt{|k|}(s - s_0)$. Then,

$$M_{DF} = \begin{bmatrix} \cosh(\theta) & \frac{\sinh(\theta)}{\sqrt{|k|}} \\ -\sqrt{|k|}\sinh(\theta) & \cosh(\theta) \end{bmatrix};$$

For a thin lens, $l \to 0$, $M_{DF} = \begin{bmatrix} 1 & 0 \\ -kl & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$ — 31

C: <u>Drift space</u>: $\frac{1}{\rho_0} = 0, \quad k = 0, \quad s = L$. Set $\theta = \sqrt{k}l = \sqrt{k}(s - s_0)$. Then,

$$M_D = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix};$$

with, $\lim_{\varphi \to 0} \frac{\sin \varphi}{\varphi}$ with $\varphi = \sqrt{k}L$ — 32

D: <u>Quadrupole Doublet</u>:

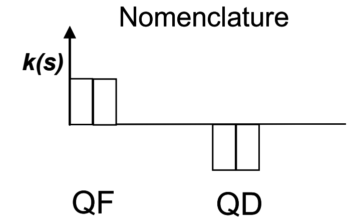
$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{f_1} & L \\ -\frac{1}{f^*} & 1 - \frac{1}{f_2} \end{bmatrix}$$

with $\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2} = \frac{L}{f^2}; f_1 = f_2 = f$ — 33

Transformation Matrices (cont.)

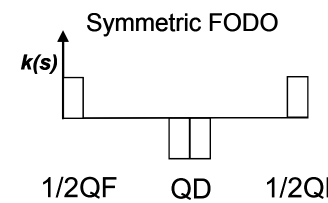
E: FODO cell - (Focusing+Drift+Defocusing+Drift)

$$M=\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{1}{f} & 1\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{1}{f} & 1\end{bmatrix}=\begin{bmatrix}1-\frac{L}{f}-\frac{L^2}{f^2} & 2L-\frac{L^2}{f}\\ -\frac{L}{f^2} & 1+\frac{L}{f}\end{bmatrix} \qquad (34)$$



FODO cell - symmetric quadrupole triplets

$$M=\begin{bmatrix}1 & 0\\ -\frac{1}{2f} & 1\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{1}{2f} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{1}{2f} & 1\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{1}{2f} & 1\end{bmatrix}=\begin{bmatrix}1-\frac{L^2}{f^2} & 2L(1-\frac{L}{f})\\ -\frac{L}{2f^2}(1+\frac{L}{f}) & 1+\frac{L^2}{2f^2}\end{bmatrix} \qquad (35)$$



We make the following point which is critical for beamline or accelerator design.

These transformation matrices enable us to follow a charged particle through a transport line/accelerator made of an arbitrary number of drift spaces, quadrupoles and bending magnets. Thus, the final transfer matrix for any system looks like,

$$M=M_n M_{n-1}\ldots M_1 \qquad (36)$$

If the sequence of elementary matrices represent components all around a circular accelerator, then we can use this matrix to investigate stability of transverse oscillations.

Stability criterion

We demand that in a synchrotron or transport line the spacing between lenses and strength of the these lenses should provide stable oscillations for passage of a charged particle beam. This implies the stability criterion is the quantity,

$$M^n \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix} \qquad \text{—— 37}$$

must remain finite for an arbitrary value of n. M is the matrix for one turn or repetition period. Then it can be shown that, the stability criterion can be met if,

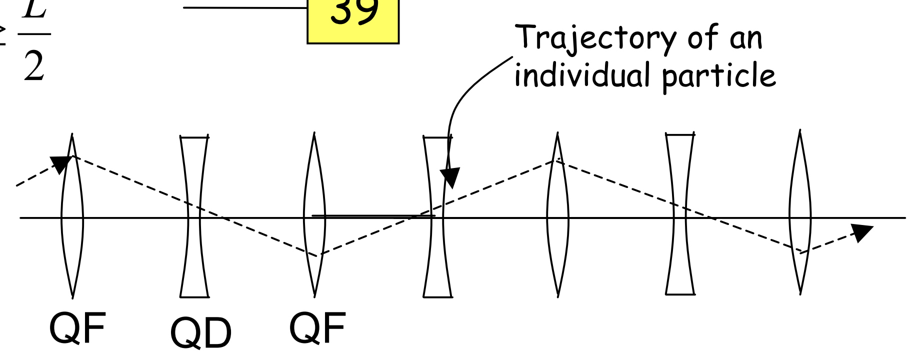
$$-1 \leq \frac{\text{Trace } M}{2} \leq 1 \qquad \text{—— 38}$$

For FODO lattice (symmetric triplets or otherwise) the stability criterion will be

$$-1 \leq 1 - \frac{L^2}{2f^2} \leq 1 \Rightarrow 0 \leq 2 - \frac{L^2}{2f^2} \leq 2$$

$$\text{or } \; f \geq \frac{L}{2} \qquad \text{—— 39}$$

<u>Betatron Oscillations:</u>
Now we can sketch oscillations of a particle traversing through a lattice described above. These oscillations are called **betatron oscillations**
Also, note that the particles do not have any slope and are on axis do not exhibit the oscillations.



Solutions for the Equation of Motion (closed form)

The equation of motion $u'' + K(s)u = 0$ has the following features,

1. The quantity K is a function of s
2. K is periodic for important class of accelerators, e.g., circular accelerators

$$\Rightarrow \quad K(s+C) = K(s)$$ C is a repeat distance or "super-period" (40)

3. Closely resembles that for simple harmonic motion

Then the general solution of such an equation can be given by

$$u(s) = A\, w(s) \cos[\psi(s) + \delta)]$$ (41)

Quantities independent of s: A, δ

w(s) is a periodic function with periodicity "C"

By substituting for *u(s)* in Eq. 22 we get, $u'' + K(s)u = 0$ (22)

$$u'' + K(s)u = -A[(2w'\psi' + w\psi'')\sin(\psi + \delta) + (w'' - w\psi'^2 + Kw)\cos(\psi + \delta) = 0$$

Equating the sine term to zero $\Rightarrow 2w'\psi' + w\psi'' = 0 = 2ww'\psi' + w^2\psi'' = (w^2\psi')'$

or $\psi' = \dfrac{C1}{w^2}$; C1 is an arbitrary constant of integration

Thus, the phase of the oscillation of particle advances according to

$$\psi(s_0 \to s_0 + C) = \Delta\psi_c = \int_{s_0}^{s_0+C} \frac{C1}{w^2(s)} ds$$ (42)

Because *w(s)* is periodic, this integral is independent of choice of s_0

Solutions (closed form) (cont.)

Now we can express the transformation matrix M in terms of "A" and $w(s)$. To do that, we rewrite the general solution as,

$$u(s) = A1\, w(s)\cos(\psi(s)) + A2\, w(s)\sin(\psi(s)) \qquad \text{43}$$

and

$$u'(s) = \left[A1 w' + \frac{A2C1}{w}\right]\cos(\psi) + \left[A2 w' - \frac{A1C1}{w}\right]\sin(\psi)$$

Now we set the initial values of u and u' at $s=s_0$ with phase angle $\psi = 0$ and solve for $A1$ and $A2$, which gives, $A1=u0/w$ and $A2=(wu_0'-u_0w')/C1$. Then the transfer matrix in terms of phase advance $\psi(s_0+C)=\Delta\psi$ looks like,

$$\begin{bmatrix} u \\ u' \end{bmatrix}_{s_0+C} = \begin{bmatrix} \cos(\Delta\psi_c) - \frac{ww'}{C1}\sin(\Delta\psi_c) & \frac{w^2}{C1}\sin(\Delta\psi_c) \\ -\frac{1+(ww'/C1)^2}{w^2/C1}\sin(\Delta\psi_c) & \cos(\Delta\psi_c) + \frac{ww'}{C1}\sin(\Delta\psi_c) \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{s_0}$$

$$= \begin{bmatrix} \cos(\Delta\psi_c) + \alpha\sin(\Delta\psi_c) & \beta\sin(\Delta\psi_c) \\ -\gamma\sin(\Delta\psi_c) & \cos(\Delta\psi_c) - \alpha\sin(\Delta\psi_c) \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{s_0} \qquad \text{44}$$

with

$$\beta(s) = \frac{w^2(s)}{C1}, \alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} = -\frac{ww'}{C1} \text{ and } \gamma(s) = \frac{1+\alpha^2}{\beta} \qquad \text{45}$$

The functions α, β and γ are called "twiss" parameters or "Courant –Snyder" parameters

Tune of an accelerator- Solutions (closed form) (cont.)

The phase advance can be written as, (after setting C1= 1)

$$\Delta\psi_c = \int_{s_0}^{s_0+C} \frac{1}{\beta(s)} ds \qquad \text{46}$$

And can be shown that $\Delta\psi_{s_1\to s_2} = \int_{s_1}^{s_2} \frac{1}{\beta(s)} ds$ and $\psi' = \frac{1}{\beta(s)}$ is the local phase advance

In particular for a circular machine, the number of betatron oscillations per turn

$$\nu = \frac{1}{2\pi} \oint \frac{1}{\beta(s)} ds \qquad \text{47}$$

is called the "tune" of the machine.

The general solution to the equation of motion becomes

$$u(s) = A\sqrt{\beta(s)} \cos[\psi(s) + \delta] \qquad \text{48}$$

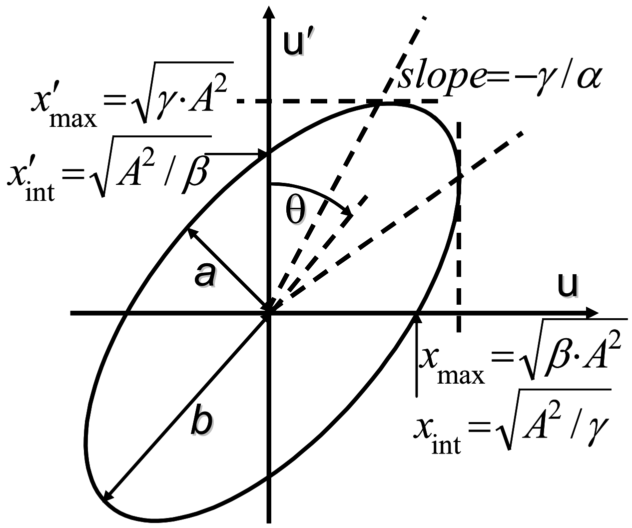
In this case one can show that

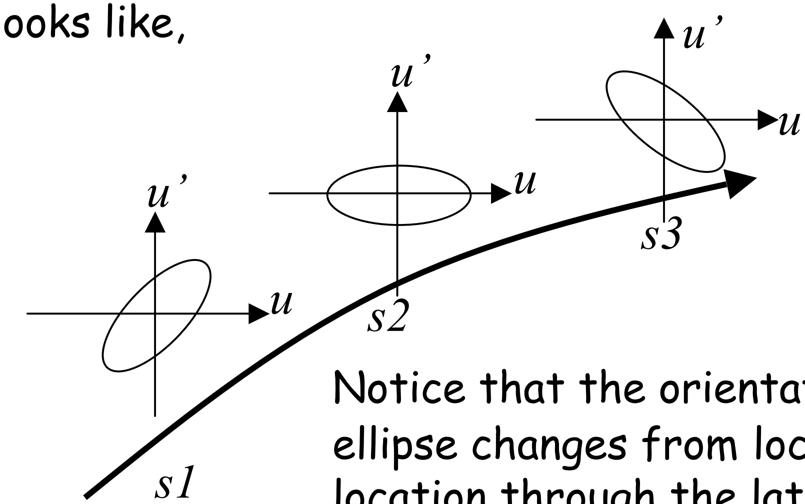
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = A^2 \Leftarrow \text{Courant -Snyder invariant} \qquad \text{49}$$

is invariant at any point in the lattice. Notice that this is an equation of ellipse in *(u,u')*-space with A^2 as its area. Hence, we conclude that the phase-space enclosed by the particle is constant (if it is not accelerted).

Solutions (closed form) (cont.)

In a circular machine, each time a particle passes a particular point in the ring its coordinates will appear as a point on the ellipse given by its amplitude and its slope at that point. Such a phase-space ellipse looks like,





Notice that the orientation of the ellipse changes from location to location through the lattice because the twiss parameters change from point to point. But the area remains constant.

Admittance: This is the phase space area associated with the largest ellipse that the accelerator will accept.

Emittance: The minimum phase space area which embeds all particles in a beam. $\Leftarrow \varepsilon$

Or, the area of the phase space ellipse spanned by the largest amplitude particle in the beam.

We have two emittances in our study of transverse dynamics: **horizontal** and **vertical** emittances.

A Few Remarks on Emittance and RMS Emittance

In reality, the transverse emittance of a beam as defined earlier, is of limited merit because it may become very large if one includes all beam particles. For example for a Gaussian beam the emittance becomes infinite. This problem can be reduced by quoting the emittance of a certain fraction of the beam, e.g. 90%, yielding $\varepsilon_{90\%}$.

One may use the rms emittance that averages over all particles with a weight given by the distance of the particles from the "center":

e.g.: $$\varepsilon^x_{rms} = \sqrt{\langle x^2\rangle \cdot \langle x'^2\rangle - \langle x \cdot x'\rangle^2}$$, with $\langle x^2\rangle = \frac{\sum_{all}(x-\bar{x})^2 \cdot c(x,x')}{\sum_{all} c(x,x')}$, $\bar{x} = \frac{\sum_{all} x \cdot c(x,x')}{\sum_{all} c(x,x')}$

$$\langle x'^2\rangle = \frac{\sum_{all}(x'-\bar{x}')^2 \cdot c(x,x')}{\sum_{all} c(x,x')}, \quad \bar{x}' = \frac{\sum_{all} x' \cdot c(x,x')}{\sum_{all} c(x,x')}, \text{ and } \langle x \cdot x'\rangle = \frac{\sum_{all}(x-\bar{x})\cdot(x'-\bar{x}')\cdot c(x,x')}{\sum_{all} c(x,x')}$$

50

This is the semi-axis-product of an ellipse. For a Gaussian distribution, the ellipse contains 39% of the beam. ($\varepsilon_{90\%} = 4.6 \cdot \varepsilon_{rms}$)

At Fermilab, we use $\varepsilon_{95\%} = 6 \cdot \varepsilon_{rms}$.

Transverse Beam Dynamics in Terms of Betatron Functions

In term of twiss parameters the trajectory of a particle can be described by,

$$u(s) = A1\sqrt{\beta(s)}\cos(\psi(s)) + A2\sqrt{\beta(s)}\sin(\psi(s))$$

Now setting $\beta = \beta_0$, at $\psi = 0$, $s = 0$ and $u(0) = u_0, u'(0) = u'_0$

and with $\psi' = \frac{1}{\beta}$ the transformation in terms of twiss parameters will be,

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos(\psi) + \alpha\sin(\psi)] & \sqrt{\beta\beta_0}\sin(\psi) \\ \sqrt{\frac{1}{\beta\beta_0}}[(\alpha_0 - \alpha)\cos(\psi) - (1 + \alpha\alpha_0)\sin(\psi)] & \sqrt{\frac{\beta_0}{\beta}}[\cos(\psi) - \alpha\sin(\psi)] \end{bmatrix} \begin{bmatrix} u_0 \\ u_0' \end{bmatrix} \quad (51)$$

The ψ is the phase advance from $s=0$ to $s=s$. In general, this can be for any $s_1 \rightarrow s_2$.

Envelope and Envelope-Equations

The quantity,

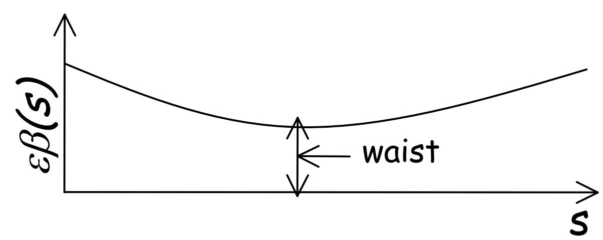
$$E(s) = u_{\max} = \sqrt{\varepsilon\beta(s)} \qquad \text{52}$$

gives the maximum transverse size of the beam at any point in an accelerator or beam transport. This is called "beam envelope".

It is important to note that $\pm\sqrt{\varepsilon\beta(s)}$ represents an envelope embedding all particles at a point in the lattice.

The beam divergence is given by,

$$u'_{\max} = \sqrt{\varepsilon}\sqrt{\frac{1+\alpha^2(s)}{\beta(s)}} = \sqrt{\varepsilon\gamma(s)} \qquad \text{53}$$



Wherever $\alpha=0$ the beam envelope $E(s)$ has a local minimum, "waist". At this point

$$E'(s) = -\alpha\sqrt{\frac{\varepsilon}{\beta}} = 0.$$

All particles in the beam with emittance ε follow the trajectory given by,

$$u(s) = \sqrt{\varepsilon\beta(s)}\cos[\psi(s)+\delta_i)]$$

Here the quantity δ_i is an arbitrary phase constant for the particle i. Substituting this in Hill's equation, and equating the $\cos(\psi+\delta_i)$ term to zero, we get,

$$2\beta\beta'' - \beta'^2 + 4K\beta^2 = 4 \qquad \text{54}$$

Envelope-Equations (Cont.)

With some mathematical rearrangements, one can write

$$E''(s) - \frac{\varepsilon^2}{E^3(s)} + KE(s) = 0$$

called the "envelope equation". 55

This plays very significant role in designing beamlines, accelerators and beam injection and extraction regions.

If we take into account the Coulomb mean field due to all particles in the beam the above equation will takes the following form,

$$E''(s) + KE(s) - \frac{\varepsilon^2}{E^3(s)} - \frac{G}{4E(s)} = 0$$

56

with $G = \frac{2I_{peak}}{[I_0/(\beta\gamma)^3]}$; I_{peak} = peak current, $I_0 = 17000\text{Amp}$; electron characteristic current

$\beta\gamma$ is relativistic factor . The above equation is called KV-envelope equation. This plays very important role in understanding beam transport of space-charge dominated beam in accelerators such as high intensity Photo-injector.

Transformation of Twiss Parameters

Let us take the initial conditions as $u=u_0$, $u'=u'_0$ and $A=\varepsilon$. Then the Courant –Snyder invariant becomes,

$$\gamma_0 u_0^{\ 2} + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^2 = \varepsilon \quad \text{—— (A)} \qquad \text{—— 57}$$

This represents the particle beam with emittance ε at $s=s0=0$. Then the transformation matrix as $s\neq 0$ is given by,

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix}$$

Solving these equations for u_0 and u'_0 and inserting them in Eq. A we get,

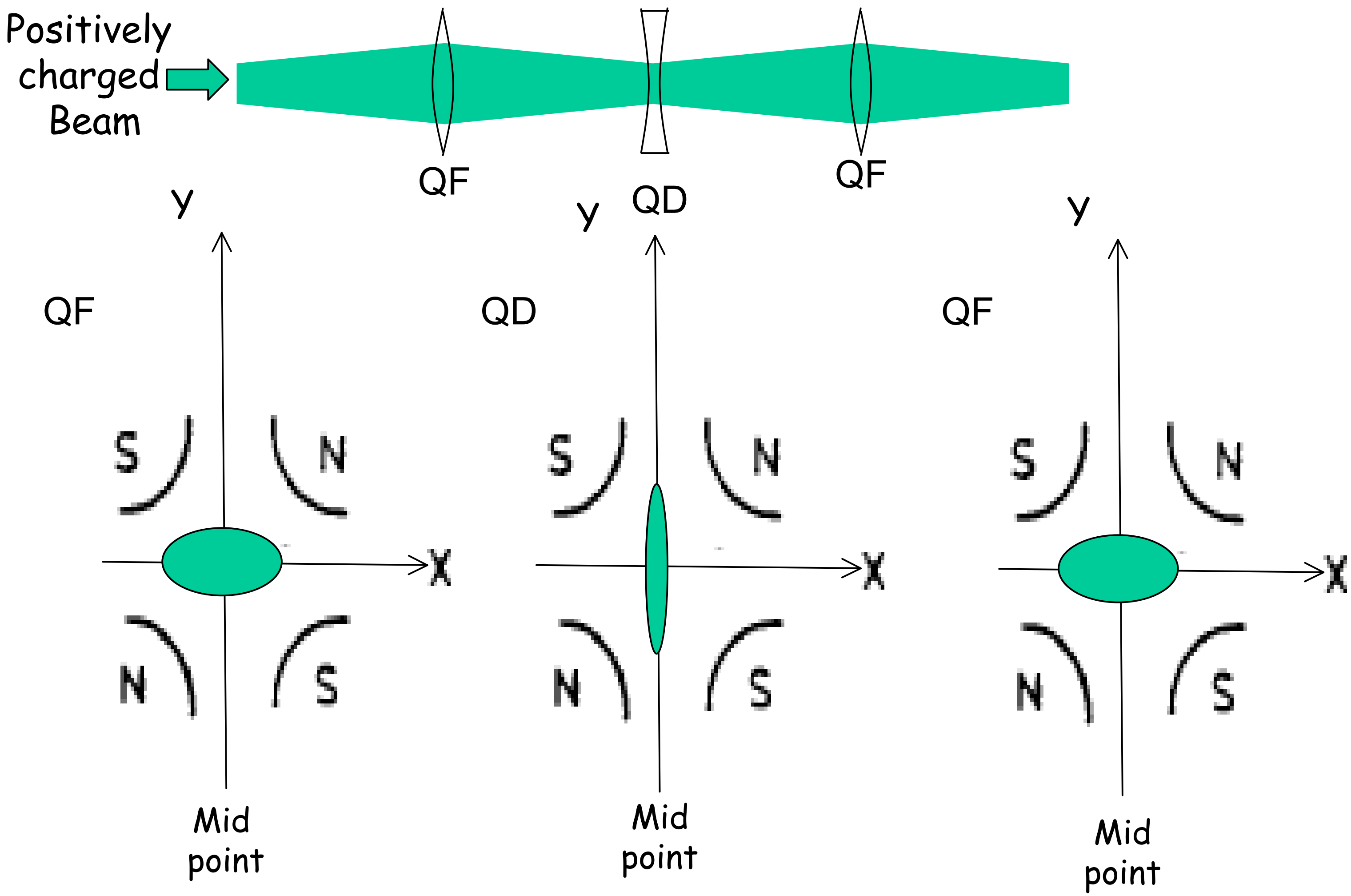
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \varepsilon \quad \text{with} \quad \begin{aligned} \beta &= C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0 \\ \alpha &= -CC'\beta_0 + (S'C + SC')\alpha_0 - SS'\gamma_0 \\ \gamma &= C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0 \end{aligned} \qquad \text{—— 58}$$

Or in matrix form the transformation of twiss parameters can be written as

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} C^2 & -2SC & S^2 \\ -CC' & (S'C + SC') & -SS \\ C'^2 & -2S'C' & S'^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{bmatrix} \qquad \text{—— 59}$$

The initial set of twiss parameters will be established from the parameters of injection region and the rest can be evaluated using the above transformation matrix.

Schematic of the beam in FODO lattice



FODO lattice, Twiss parameters, Envelope

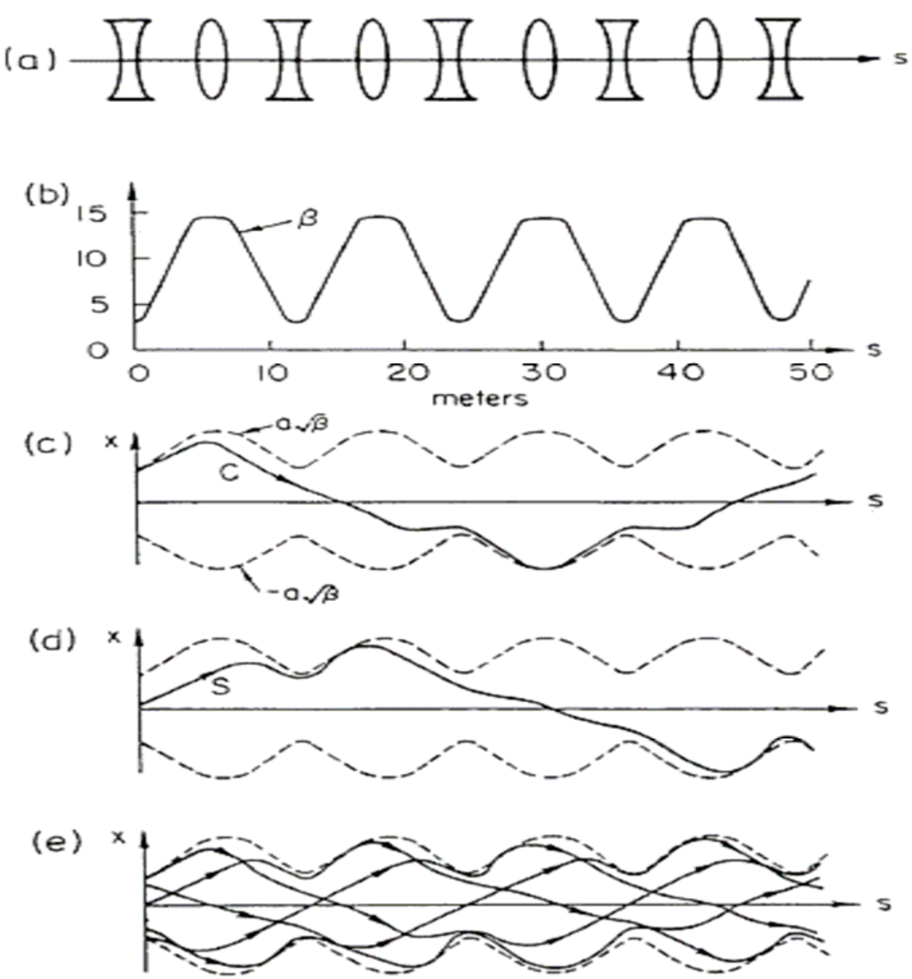


Figure 28: **a)** A regular FODO lattice of focusing and defocusing lenses **b)** Beta-function. **c)** Cosine-like trajectory for $s = 0$. **d)** Sine-like trajectory for $s = 0$. **e)** One trajectory on several successive revolutions.
(According to M. Sands, The Physics of Electron Storage Rings, SLAC–121).